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A dynamic boundary value problem arising in the ecology of mangroves ☆

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Abstract

We consider an evolution model describing the vertical movement of water and salt in a domain splitted in two parts: a water reservoir and a saturated porous medium below it, in which a continuous extraction of fresh water takes place (by the roots of mangroves). The problem is formulated in terms of a coupled system of partial differential equations for the salt concentration and the water flow in the porous medium, with a dynamic boundary condition which connects both subdomains.

We study the existence and uniqueness of solutions, the stability of the trivial steady state solution, and the conditions for the root zone to reach, in finite time, the threshold value of salt concentration under which mangroves may live.

Keywords: Dynamic boundary condition; System of partial differential equations; Existence; Uniqueness; Dead core

1. Introduction

Mangrove forests or swamps can be found on low, muddy, tropical coastal areas around the world. Mangroves are woody plants that form the dominant vegetation of mangrove forests. They are characterized by their ability to tolerate regular inundation by tidal water with salt concentration c_{w} close to that of sea water (see, for example, [18]). The mangrove roots take up fresh water from the saline soil and leave behind most of the salt, resulting in a net flow of water downward from the soil surface, which carries salt with it. As pointed out by Passioura et al. [25], in the absence of lateral flow, the steady state salinity profile in the root zone must be such that the salinity around the roots is higher than c_{w}, and that the concentration gradient is large enough so that the advective downward flow of salt is balanced by the diffusive flow of salt back up to the surface. In [25] the authors presented steady state equations governing the flow of salt and uptake of water in the root zone, assuming that there is an upper limit c_{c} to the salt concentration at which roots can take up water, and that the rate of uptake of water is proportional to the difference between the local concentration c and the assumed upper limit c_{c}. They also assumed that the root zone is unbounded, and that the constant of proportionality for root water uptake is independent of depth through the soil. In [12], the model was extended in two important ways. First, considering more general root water uptake functions and second, limiting the root zone to a bounded domain. The authors proved mathematical properties such as the existence and uniqueness of solutions of the

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evolution and steady state problems, the conditions under which the threshold level of salt concentration is attained, and others. In [12], it is assumed that tides, or other sources of fresh or not too saline water, renew the water on the soil–water interface allowing to prescribe the salt concentration at this boundary (Dirichlet boundary data). Although this is the usual situation in which mangroves live, in this article we shall focus in the situation in which the inflow of fresh or sea water is impeded. In this situation, the continuous extraction of fresh water by the roots of mangroves drives the ecosystem to a complete salinization and, henceforth, to death. This work is motivated by the occurrences observed at Ciénaga Grande de Santa Marta, Colombia. As reported by Botero [8] (see also [28]), the construction of a highway along the shore in the 1950s obstructed the natural circulation of water between both parts of the road (Caribbean sea and lagoon). In addition, in the 1970s, inflow of fresh water from the river Magdalena was reduced due to the construction of smaller roads and flooding control dikes. These changes caused a hypersalinization of water and soil, which resulted in approximately 70% mangrove mortality (about 360 Km^2 of mangrove forests), see [8,17]. Although other causes, like evaporation or sedimentation, may have had an important contribution to the salinization of the Ciénaga, we shall keep our attention in the mechanisms of mangroves and their influence in this process.

The main mathematical difficulty of this model when compared with that studied in [12] is that the closure of the natural system, the lagoon, implies a new type of boundary condition in the water–soil interface, which is no longer of Dirichlet type. Balance equations for salt and water content lead to a *dynamical boundary condition* at such interface, i.e., a boundary condition involving the time derivative of the solution. Although not too widely considered in the literature, dynamic boundary conditions date back at least to 1901 in the context of heat transfer [26]. Since then, they have been studied in many applied investigations in several disciplines like Stefan problems [29,32], fluid dynamics [16], diffusion in porous medium [27,15], mathematical biology [14] or semiconductor devices [30]. From a more abstract point of view the reader is referenced to, among others [10,23,19,11,13,1,2,7].

Apart from the mathematical technical details, one of the main features of the dynamic boundary condition when compared to the Dirichlet boundary condition is the elimination of the boundary layer the latter creates in a neighborhood of the water–soil interface, layer in which the salt concentration keeps well below the threshold salinity level. Thus, this new model allows us to describe the situation in which a continuous increase of fresh water uptake by the roots of mangroves drives the ecosystem to a complete salinization.

The outline of the paper is the following: in Section 2 we formulate the mathematical model. We assume that mangroves roots are situated in a porous medium in the top of which a water reservoir keeps the soil saturated. As in [12], coupled partial differential equations for salt concentration and water discharge are considered in the porous medium. Above it, in the water reservoir, balance laws for salt and water are formulated. The assumption of homogeneous salt concentration in the water reservoir leads to a dynamic boundary condition in the water–soil interface. In Section 3 we state our hypothesis and formulate our main results on existence and uniqueness of solutions of the evolution problem, as well as the convergence of this solution to the steady state solution. We also study the conditions under which the complete salinization of the root zone is attained in finite time (dead core). Finally, in Section 4 we prove our assertions.

2. The mathematical model

In this section we formulate the mathematical model which describes the salt and water movement in the water–soil system. We consider the case where the mangroves are present in the horizontal x, y plane, with an homogeneous porous medium located below this plane and a water reservoir above it. The porous medium is characterized by a constant porosity θ, indicating that we are assuming the mangroves roots to be homogenized throughout the porous medium, without affecting its properties. Assuming further that the hydrodynamic dispersion tensor, $\mathbf{D}$, is constant and isotropic, i.e. neglecting the velocity dependence in the mechanical dispersion, we find for the salt concentration the equation, see [6],

$$\theta \frac{\partial c}{\partial t} + \operatorname{div}(c\mathbf{q} - \theta \mathbf{D} \nabla c) = 0, \tag{1}$$

where the vector $\mathbf{q}$ denotes the specific discharge of the fluid, $\mathbf{D} = D\mathbf{I}$, $\mathbf{I}$ is the identity matrix and t denotes time. We also have a fluid balance in the porous medium. Disregarding density variations in the mass balance equation of the fluid, we obtain a fluid volume balance expressed by

$$\mathrm{div}\mathbf{q} + S = 0, \tag{2}$$

where S is the volume of water taken up by the roots per unit volume of porous material per unit time. If the mangroves are uniformly distributed throughout the x, y-plane and there is no lateral fluid flow, we may consider the problem as one-dimensional in the vertical direction. If the z-axis is positive when pointing downwards, the flow domain is characterized by the interval $0 < z < H < \infty$. In the one-dimensional setting Eqs. (1) and (2) become

$$\theta c_t + (cq - \theta D c_z)_z = 0, \tag{3}$$

$$q_z + S = 0, \quad \text{in } (0, H) \times (0, T). \tag{4}$$

For S, we assume to have the form

$$S := \begin{cases} s(z)\left(1 - \dfrac{c}{c_{\mathrm{c}}}\right)^p & \text{for } 0 \leqslant c \leqslant c_{\mathrm{c}}, \\ 0 & \text{for } c > c_{\mathrm{c}}, \end{cases} \tag{5}$$

where c_{c} is the upper limit of salt concentration at which mangroves may uptake water, $p > 0$ and $s(z)$ is determined by the root distribution as a function of the depth z below the soil surface. This root distribution function will be non-negative, and non-increasing with z. We shall keep in mind the following characteristic example: we assume that the function s is a positive constant, s_0/z_*, above a certain depth z_*, and zero below that depth, i.e.

$$s(z) = s_0/z_* \quad \text{if } 0 \leqslant z \leqslant z_* \quad \text{and} \quad s(z) = 0 \quad \text{if } z_* < z \leqslant H. \tag{6}$$

The quantity s_0 is the total amount of root water uptake in the profile with no salt present, in volume per unit surface per unit time, i.e. the *transpiration rate* of the mangrove plants in the absence of salinity. On the bottom of the porous medium domain, we assume no flux boundary conditions, resulting in

$$q(H, t) = c_z(H, t) = 0 \quad \text{for } t \in (0, T). \tag{7}$$

On the water–soil interface we prescribe a boundary condition which is deduced from conservation laws for salt and water in the whole system water–soil. We assume that salt concentration in the water domain, C, remains uniformly distributed in space. This approximation is justified when assuming a much faster mixing of the salt in the reservoir than in the porous medium. Then, the average height level of the water reservoir, W, and C are functions that only depend on time. We further consider, based on a continuity assumption

$$C(t) = c(0, t) \quad \text{for } t \in (0, T). \tag{8}$$

Then we have

- The salt balance. Assuming that the total amount of salt in the system water–soil remains constant, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(CW + \int_0^H \theta c\right) = 0 \quad \text{in } (0, T).$$

  Therefore, from Eq. (3) and the boundary condition (7),

$$\frac{\mathrm{d}(CW)}{\mathrm{d}t} = c(0, \cdot)q(0, \cdot) - \theta D c_z(0, \cdot) \quad \text{in } (0, T). \tag{9}$$

- The fluid balance, which asserts that the amount of water taken up from the soil by the roots of mangroves is replaced by water from the reservoir:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -q(0, \cdot) \quad \text{in } (0, T). \tag{10}$$

Combining (8)–(10) we deduce

$$W(t)c_t(0,t) = \theta D c_z(0,t) \quad \text{for } t \in (0,T), \tag{11}$$

which is the dynamic boundary condition for the soil–water interface. Finally, we add to this formulation given initial distributions of salt concentration, $c(\cdot,0) = c_0$ in $(0,H)$, and of water reservoir height level, $W(0) = W_0$.

We recast the above formulation in an appropriate dimensionless form introducing the following variables, unknowns and parameters:

$$\tilde{t} := Dt/z_*^2, \quad x := z/z_*, \quad u := c/c_c, \quad \tilde{q} = qz_*/D\theta,$$

$$w = W/\theta z_*, \quad \tilde{s}(x) := z_* s(Hx)/s_0, \quad d := H/z_*, \quad m := s_0 z_*/D\theta$$

and we define $f(x,u) := S(Hx, c_c u)$, with $f : [0,d] \times [0,1] \to \mathbb{R}_+$ given by

$$f(x,\sigma) := \tilde{s}(x)(1-\sigma)_+^p, \tag{12}$$

with $p > 0$ and

$$\tilde{s}(x) = 1 \quad \text{if } 0 \leqslant x \leqslant 1 \quad \text{and} \quad \tilde{s}(x) = 0 \quad \text{if } 1 < x \leqslant d. \tag{13}$$

With the above changes we are led to the following problem (omitting tildes): find $u : \bar{Q}_T \to [0,1]$, $q : \bar{Q}_T \to \mathbb{R}$ and $W : [0,T] \to \mathbb{R}$ such that

$$u_t + (uq - u_x)_x = 0, \tag{14}$$

$$q_x + mf(\cdot,u) = 0 \quad \text{in } Q_T = I \times (0,T), \text{ with } I = (0,d), \tag{15}$$

$$w'(t) + q(0,t) = 0 \quad \text{for } t \in (0,T),$$

subject to the boundary and initial conditions

$$w(t)u_t(0,t) = u_x(0,t), \tag{16}$$

$$u_x(d,t) = q(d,t) = 0 \quad \text{for } t \in (0,T), \tag{17}$$

$$u(\cdot,0) = u_0 \quad \text{in } I, \qquad w(0) = w_0. \tag{18}$$

Remark 1. In the recasting of our model there appeared a constant capturing all the important physical parameters, the mangrove's number:

$$m := s_0 z_*/D\theta. \tag{19}$$

Using [25] and [24] as a reference we find the following values for the physical constants: $D = 7 \times 10^{-5}\,\mathrm{m}^2/\mathrm{day}$, $\theta = 0.5$, and $s_0 = 1\,\ell\mathrm{m}^{-2}\mathrm{day}^{-1}$. Taking z^* in the range 0.2–0.5 m, this implies a time scale in the range 2–10 yr and $m \in (6,15)$.

3. Main results

We shall refer to problem (14)–(18), as to Problem P, for which we assume the following hypothesis:

H_1. The function $f : \bar{I} \times [0,1] \to \mathbb{R}$, with $I = (0,d)$ and $d \geqslant 1$, satisfies

$$f \in L^\infty(I; C([0,1])), \quad |f| \leqslant 1,$$

$$f(\cdot,s) \text{ is non-increasing in } \bar{I} \text{ and } f(d,s) \geqslant 0 \text{ for all } s \in [0,1],$$

$$f(x,\cdot) \text{ is non-increasing in } [0,1] \text{ and } f(x,1) = 0 \text{ for a.e. } x \in I.$$

Note that, in particular, $f \geqslant 0$ in $\bar{I} \times [0,1]$.

H_2. The initial data posses the regularity

$$u_0 \in H^1(I) \quad \text{with } 0 \leqslant u_0 \leqslant 1 \text{ in } I.$$

H_3. The function w is a positive constant. The number m is positive. We set $w = m = 1$.

Remark 2. The assumption w (or the dimensional W) constant in H_3 has a reasonable range of validity. From (4), (5), (10) and the mean value theorem we infer

$$W(t) = W_0 - \int_0^t q(0,\tau)\,\mathrm{d}\tau = W_0 - ts_0\left(1 - \frac{\bar{c}}{c_\mathrm{c}}\right)^p \quad \text{for some } \bar{c} \in (0, c_\mathrm{c}).$$

Set s_0 as in Remark 1 and $p = 1$. A lower limit for $\bar{c}$ is sea water salt concentration $c_\mathrm{w} \sim 0.5c_\mathrm{c}$. Then W_0 must be much greater than the 15 cm that the lagoon will decrease per year while keeping the sea water salt concentration. For a value of $c = 0.9c_\mathrm{c}$ the decrease of the height level is of about 3 cm per year.

Remark 3. Since the numbers m and w do not play any essential role in the results we prove in this work, we set $m = w = 1$ for clarity.

Under Hypothesis H_1–H_3 we cannot expect the existence of classical solutions. We then introduce the notion of solution we shall work with.

Definition 1. We say that (u, q) is a *strong solution* of Problem P if $u : \bar{Q}_T \to [0, 1]$ and $q : \bar{Q}_T \to \mathbb{R}$ satisfy the following properties:

(1) For any $r \in (0, \infty)$,

$$u \in W^{1,r}(0, T; L^r(I)) \cap L^r(0, T; W^{2,r}(I)) \cap C((0, T]; C(\bar{I})),$$

$$q \in C((0, T]; \mathscr{W})$$

with $\mathscr{W} := \{\varphi \in W^{1,\infty}(I) : \varphi(d) = 0\}$.

(2) The differential Eqs. (14) and (15) and the boundary conditions (16) and (17) are satisfied almost everywhere. The initial distribution is satisfied in the sense

$$\lim_{t\to 0} \|u(\cdot, t) - u_0\|_{L^2(I)} = 0.$$

We prove the following result on existence and regularity of solutions.

Theorem 1. *Assume* H_1–H_3. *Then there exists a strong solution of Problem* P *satisfying*

$$u \geqslant u_m := \min_{\bar{I}} u_0 \quad \textit{a.e. in } Q_T. \tag{20}$$

In addition, if for some $p > 0$

$$f \in C^p(\bar{I} \times [0, 1]) \quad \textit{and} \quad u_0 \in C^{2+p}(\bar{I}) \tag{21}$$

and if u_0 *satisfies the following compatibility condition*:

$$u_0'(0) + u_0'(0)\int_0^1 f(x, u_0(x))\,\mathrm{d}x - u_0''(0) = f(0, u_0(0))u_0(0), \tag{22}$$

then $u \in C^{1+p,2+p}(\bar{Q}_T)$ *and* $q \in C^{1+p,1+p}(\bar{Q}_T)$.

We prove uniqueness of solution for $f(x, \cdot)$ being Lipschitz continuous in [0, 1]. For more general functions, we show that uniqueness of solution holds true under an additional condition on the component u. In Proposition 1 we give an example in which solutions of Problem P satisfy such condition.

Theorem 2. *Let* (u_1, q_1) *and* (u_2, q_2) *be two strong solutions of Problem* P *and let* H_1–H_3 *be satisfied. If either*

$$f(x,\cdot) \text{ is Lipschitz continuous in } [0,1] \text{ for almost all } x \in \Omega, \tag{23}$$

or anyone of the solutions satisfies

$$u(x,t) > \int_0^x |u_x(y,t)|\,\mathrm{d}y \quad \text{a.e. in } Q_T, \tag{24}$$

then $(u_1, q_1) = (u_2, q_2)$ *a.e. in* Q_T.

Proposition 1. *Assume* H_1–H_3 *and* (21)–(22), *and let* (u, q) *be a solution of Problem* P. *Suppose that* u_0 *satisfies* $u_{0x} \leqslant L$ *in* I *and*

$$f(\cdot, u_m) \leqslant L < u_m - \tfrac{1}{2} \tag{25}$$

for some positive constant, L, *with* u_m *given by* (20). *Assume*

$$f(\cdot, u) + u f_u(\cdot, u) < 0 \quad \text{in } Q_T. \tag{26}$$

Then condition (24) *is satisfied.*

Remark 4. In particular, if $f(x,\sigma) = s(x)(1-\sigma)^p$, with s smooth, and $u_0 \in C^{2+p}(\bar{I})$ satisfies $u_{0x} \leqslant (1-u_m)^p < u_m - \frac{1}{2}$, then condition (24) is satisfied. Actually, the smoothness requirement on s may be dropped by using an approximation argument.

One important effect of the dynamic boundary condition when compared to the Dirichlet boundary condition at the boundary water–soil is the elimination of the boundary layer the latter creates. It is straightforward to prove that the unique solution of the steady state problem corresponding to Problem P, i.e., functions $U \in H^1(I)$ and $Q \in \mathscr{W}$ satisfying

$$(QU - U_x)_x = Q_x + f(\cdot, U) = 0 \quad \text{in } I,$$

$$U_x(0) = U_x(d) = 0,$$

is the trivial solution $(U, Q) = (1, 0)$. Regarding the asymptotic convergence of solutions of Problem P to this trivial solution when $t \to \infty$, we have the following result.

Theorem 3. *Assume* H_1–H_3 *and* $u_m > 0$, *and let* (u, q) *be a strong solution of Problem* P. *Then*

$$(u, q) \to (1, 0) \quad \text{in } L^2(I) \quad \text{and} \quad u(0,t) \to 1 \quad \text{pointwise as } t \to \infty.$$

We finally state a result on the existence of a dead core for solutions of Problem P, i.e., sets where the threshold salinization $u = 1$ is attained in finite time. The proof of this result, which is of local nature, i.e., independent of the boundary data, can be found in [12]. First, we introduce some notation. For any $t \in (0, T)$ we consider the parabola of vertex (x_0, t),

$$\mathscr{P}(t) := \{(x,\tau) : |x - x_0| < (\tau - t)^\nu,\ \tau \in (t, T)\},$$

with $0 < \nu < 1$ and $x_0 \in I$ such that $T^\nu < x_0 < 1 - T^\nu$, implying $\mathscr{P}(t) \subset Q_T$ for all $t \in (0, T)$. We define the *local energy functions*

$$E(t) := \int_{\mathscr{P}(t)} |u_x|^2 \,\mathrm{d}x\,\mathrm{d}\tau \quad \text{and} \quad C(t) := \int_{\mathscr{P}(t)} (1-u)^{p+1} \,\mathrm{d}x\,\mathrm{d}\tau. \tag{27}$$

In [12] we proved the following theorem using the techniques introduced in [3,4].

Theorem 4. *Suppose there exist constants s_0 and s_1 such that*

$$0 < s_0\sigma^{p+1} \leqslant \sigma f(\cdot, 1-\sigma) \leqslant s_1\sigma^{p+1} \quad \text{for } \sigma \in [0,1], \tag{28}$$

in $\mathscr{P}(t)$ for a.e. $t \in (0,T)$, with $p \in (0,1)$ and $s_0 > s_1/2$, and let (u,q) be a strong solution of Problem P. *Then there exists a positive constant M such that if $E(0) + C(0) \leqslant M$ then $u \equiv 1$ in $\mathscr{P}(t^*)$ for some $t^* \in (0,T)$.*

Let us finish this section with a remark on the assumptions of Theorem 4. First, if function f is given by $f(x,\sigma) = s(x)(1-\sigma)^p$, with s given by (13) then (28) is trivially satisfied in the region where $s > 0$ (root zone). Regarding the bound of the initial energy, we have that testing the first equation of (14) with $1-u$ and using the Eq. (15) we obtain

$$2E(0) + C(0) \leqslant \int_I (1-u_0)^2 + (1-u_0(0))^2 \left(1 + \int_{Q_T} f(\cdot, u)\right). \tag{29}$$

Therefore, if the initial datum is close enough to one then the initial energy bound is satisfied. Combining Theorems 3 and 4 we deduce the following corollary.

Corollary 1. *Let (u,q) be a strong solution of Problem* P *in Q_T, for T large enough. Under the conditions of Theorems* 3 *and* 4 *there exist $T_0, t^* > 0$ such that $u \equiv 1$ in $\mathscr{P}(t^*)$, for some $t^* \in (T_0, T)$.*

Or, in other words, the threshold value of salt concentration is attained in any compact set contained in the root zone in finite time.

4. Proofs

Proof of Theorem 1. We first prove the existence of weak solution of a time discretization of Problem P. Since, a priori, the component u of solutions to approximated problems will not necessarily satisfy $0 \leqslant u \leqslant 1$, we extend f by $\bar{f}$ as $\bar{f}(x,\sigma) = 0$ if $\sigma > 1$, $\bar{f}(x,\sigma) = f(x,\sigma)$ if $0 \leqslant \sigma \leqslant 1$ and $\bar{f}(x,\sigma) = f(x,0)$ if $\sigma < 0$. We denote the corresponding problem by Problem $\bar{\text{P}}$.

Lemma 1. *For $\tilde{u} \in H^1(I)$, and $\tau > 0$ small enough, there exists a solution $(u,q) \in W^{2,r}(I) \times \mathscr{W}$, with $r < \infty$, of*

$$u + \tau(uq - u_x)_x = \tilde{u} \quad \text{a.e. in } I, \tag{30}$$

$$q_x + \bar{f}(\cdot, u) = 0 \quad \text{a.e. in } I, \tag{31}$$

$$u(0) = \tilde{u}(0) + \tau u_x(0), \quad u_x(d) = 0. \tag{32}$$

Proof. We introduce the set $K = \{v \in \mathscr{W}, \|v\|_{W^{1,\infty}} \leqslant \rho\}$ for some $\rho > 0$ to be fixed. It is clear that K is convex and weakly compact in the star topology of $W^{1,\infty}(I)$. For $\hat{q} \in K$, we define the map

$$S(\hat{q})(x) := \int_x^d \bar{f}(s, u(s))\,\mathrm{d}s$$

with $u \in H^1(I)$ solution of

$$\int_I (u-\tilde{u})\varphi + \tau\int_I u_x\varphi_x + \tau\int_I (u\hat{q})_x\varphi + (u(0) - \tilde{u}(0))\varphi(0) = 0 \tag{33}$$

for any $\varphi \in H^1(I)$. The existence of a unique solution of (33) is guaranteed by the Theorem of Lax–Milgram (see, for instance, [9]). In addition, we have

$$\|u_{xx}\|_{L^2} \leqslant \frac{1}{\tau}\|u - \tilde{u}\|_{L^2} + \|u\|_{L^2}\|\bar{f}\|_{L^\infty} + \|u_x\|_{L^2}\|q\|_{L^\infty}, \tag{34}$$

i.e., $u \in H^2(I)$. Since $\tilde{u} \in H^1(I) \subset C(\bar{I})$, a boot-strap argument allows us to deduce $u \in W^{2,r}(I)$, for any $r < \infty$. A standard argument allows us to conclude that u satisfies (30) and (32) (with $\hat{q}$ replaced by q).

Observe that a fixed point of S is a solution of (30)–(32). We prove the existence of such a fixed point using a theorem by [5], for which we need to show: (i) $S(K) \subset K$ and (ii) S is weakly–weakly continuous in the star topology of $W^{1,\infty}(I)$. Showing $S(K) \subset K$ is straightforward since for any $\hat{q} \in K$, $\|S(\hat{q})\|_{W^{1,\infty}} \leqslant 2\|\bar{f}\|_{L^\infty} =: \rho$.

To prove the weak continuity, (ii), we consider a sequence $\hat{q}_j$ and a function $\hat{q}$ in K such that $\hat{q}_j \to \hat{q}$ weakly star in $W^{1,\infty}(I)$. Let u_j and u be the corresponding solutions of problem (33). Taking $\varphi = u_j$ in (33) we obtain, after using Schwarz's inequality,

$$u_j(0)^2 + (1 - \tau\|\hat{q}_j\|^2_{L^\infty} - 2\tau\|\hat{q}_{jx}\|_{L^\infty})\|u_j\|^2_{L^2} + \tau\|u_{jx}\|^2_{L^2} \leqslant \|\tilde{u}\|^2_{L^2} + \tilde{u}(0)^2.$$

For τ small enough and independent of j we get $E_j + \tau\|u_{jx}\|^2_{L^2} \leqslant c$, with c independent of τ and j, and with

$$E_j = u_j(0)^2 + \|u_j\|^2_{L^2}. \tag{35}$$

Therefore, we obtained a uniform bound which allows us to extract a subsequence of u_j (not relabelled) such that $u_j \to v$ weakly in $H^1(I)$, for some $v \in H^1(I)$. Since the embedding $H^1(I) \subset C(\bar{I})$ is compact, extracting a new subsequence if necessary we have $u_j \to v$ uniformly in $C(\bar{I})$. Next we show that, actually, $v = u$. All the terms in (33) corresponding to $(u_j, \hat{q}_j)$ are well defined in the limit $j \to \infty$. For instance,

$$\int_I (u_j\hat{q}_j)_x\varphi = \int_I u_{jx}\hat{q}_j\varphi + \int_I u_j\hat{q}_{jx}\varphi \to \int_I (v\hat{q})_x\varphi,$$

due to the convergences $u_j \to u$ weakly in $H^1(I)$ and uniformly in $C(\bar{I})$, and $\hat{q}_j \to \hat{q}$ weakly star in $W^{1,\infty}(I)$ and uniformly in $C(\bar{I})$ (by compact embedding, again). Then, by the uniqueness of solution of problem (33) we deduce $v = u$. Hence,

$$S(\hat{q}_j)(x) = \int_x^d \bar{f}(s, u_j(s))\,\mathrm{d}s \to \int_x^d \bar{f}(s, u(s))\,\mathrm{d}s = S(\hat{q})(x),$$

uniformly in $C^1(\bar{I})$ and, in particular, weakly star in $W^{1,\infty}(I)$. Therefore, (ii) is proven and the existence of a fixed point deduced. $\square$

We now construct piecewise constant in time approximations of solutions of Problem $\bar{\mathrm{P}}$. Let $(0, T] = \bigcup_{k=1}^K ((k-1)\tau, k\tau]$, with $\tau = T/K$ and $K \in \mathbb{N}$. For $k = 1, \dots, K$, define recursively (u_k, q_k) as the solution of problem (30)–(32) with $\tilde{u} = u_{k-1}$, $u = u_k$ and $q = q_k$. Let the initialization of this recursion be the initial data of Problem $\bar{\mathrm{P}}$, u_0. We define the following piecewise constant in time functions: $u^{(\tau)}(x,t) = u_k(x)$, $q^{(\tau)}(x,t) = q_k(x)$,

$$\partial_t^{(\tau)} u^{(\tau)}(x,t) = \frac{u_k(x) - u_{k-1}(x)}{\tau}, \quad E^{(\tau)}(t) = \tfrac{1}{2}\left(|u^{(\tau)}(0,t)|^2 + \int_I |u^{(\tau)}|^2\right)$$

if $x \in I$, $t \in ((k-1)\tau, k\tau]$, for $k = 1, \dots, K$.

Lemma 2. *As $\tau \to 0$ there exist a subsequence of $(u^{(\tau)}, q^{(\tau)})$ (not relabelled) such that*

$$u^{(\tau)} \to u \quad \textit{weakly star-weakly in } L^\infty(0,T; H^1(I)), \tag{36}$$

$$\partial_t^{(\tau)} u^{(\tau)} \to \partial_t u \quad \textit{weakly in } L^2(Q_T), \tag{37}$$

$$\partial_t^{(\tau)} u^{(\tau)}(0,\cdot) \to \partial_t u(0,\cdot) \quad \textit{weakly in } L^2(0,T), \tag{38}$$

$$u^{(\tau)} \to u \quad \textit{weakly in } L^2(0,T; H^2(I)), \tag{39}$$

$$u^{(\tau)} \to u \quad \textit{uniformly in } C((0,T]; C(\bar{I})), \tag{40}$$

$$q^{(\tau)} \to q \quad \textit{uniformly-strongly in } C((0,T]; \mathscr{W}). \tag{41}$$

Proof. Replacing in (30) functions u, q and $\tilde{u}$ by u_k, q_k and u_{k-1}, respectively, and using $\varphi = u_k$ in the weak formulation (33) (with $\hat{q} = q$), we obtain, after using the inequalities of Schwarz and $x(x-y) \geqslant (x^2 - y^2)/2$,

$$E_k + \tau \|u_{kx}\|^2_{L^2} \leqslant E_{k-1} + \tau c_f E_k$$

for

$$E_k = \tfrac{1}{2}\left(u_k(0)^2 + \int_I u_k^2\right)$$

and with $c_f := \|\bar{f}\|^2_{L^\infty} + \|\bar{f}\|_{L^\infty}$. Then, from the Gronwall's discrete inequality and $k\tau \leqslant K$, we deduce $E_k \leqslant cE_0$, for $k = 1, \ldots, K$, and for some constant, c, independent of τ. Therefore,

$$\frac{E_k - E_{k-1}}{\tau} + \|u_{kx}\|^2_{L^2} \leqslant cc_f E_0.$$

Integrating in $(0, t)$ for any $t \in (0, T)$, we obtain

$$E^{(\tau)}(t) + \int_{Q_t} |u_x^{(\tau)}|^2 \leqslant cc_f E_0,$$

which gives a uniform estimate for $u^{(\tau)}$ in the norm of $L^2(0, T; H^1(I)) \cap L^\infty(0, T; L^2(I))$. On the other hand, from (31) we obtain $\|q_k\|_{W^{1,\infty}} \leqslant \|\bar{f}\|_{L^\infty}$, which implies the uniform bound

$$\|q^{(\tau)}\|_{L^\infty(W^{1,\infty})} \leqslant \|\bar{f}\|_{L^\infty}.$$

We now choose $\varphi = (u_k - u_{k-1})/\tau$ in (33) (with $\hat{q} = q$). We get

$$\int_I \left|\frac{u_k - u_{k-1}}{\tau}\right|^2 + \int_I u_{kx}\left(\frac{u_k - u_{k-1}}{\tau}\right)_x + \int_I (u_k q_k)_x \frac{u_k - u_{k-1}}{\tau} + \left|\frac{u_k(0) - u_{k-1}(0)}{\tau}\right|^2 = 0.$$

Using again the inequality $x(x-y) \geqslant (x^2 - y^2)/2$, we obtain

$$\int_I u_{kx}\left(\frac{u_k - u_{k-1}}{\tau}\right)_x \geqslant \frac{1}{2\tau}\int_I (|u_{kx}|^2 - |u_{(k-1)x}|^2)$$

and therefore

$$\begin{aligned}&\int_I \left|\frac{u_k - u_{k-1}}{\tau}\right|^2 + \frac{1}{2\tau}\int_I (|u_{kx}|^2 - |u_{(k-1)x}|^2) + \int_I (u_k q_k)_x \frac{u_k - u_{k-1}}{\tau}\\ &\quad + \left|\frac{u_k(0) - u_{k-1}(0)}{\tau}\right|^2 \leqslant 0.\end{aligned}$$

Integrating in $((k-1)\tau, k\tau)$ and adding from $k = 1$ to K leads to

$$\begin{aligned}&\frac{1}{2}\int_I |u_x^{(\tau)}|^2(T, \cdot) + \int_{Q_T} |\partial_t^{(\tau)} u^{(\tau)}|^2 + \int_0^T |\partial_t^{(\tau)} u^{(\tau)}(0, \cdot)|^2\\ &\quad \leqslant \frac{1}{2}\int_I |u_{0x}|^2 - \int_{Q_T} (u^{(\tau)} q^{(\tau)})_x \partial_t^{(\tau)} u^{(\tau)}.\end{aligned}$$

Using Hölder's inequality we deduce

$$\begin{aligned}&\|u_x^{(\tau)}\|^2_{L^\infty(L^2)} + \|\partial_t^{(\tau)} u^{(\tau)}\|^2_{L^2(L^2)} + \|\partial_t^{(\tau)} u^{(\tau)}(0, \cdot)\|^2_{L^2(0,T)}\\ &\quad \leqslant c(\|u_0\|^2_{H^1} + \|q^{(\tau)}\|^2_{W^{1,\infty}} \|u^{(\tau)}\|^2_{L^2(H^1)}),\end{aligned}$$

i.e., additional uniform bounds for

$$u^{(\tau)} \quad \text{in } L^\infty(0, T; H^1(I)),$$

$$\partial_t^{(\tau)} u^{(\tau)} \quad \text{in } L^2(Q_T), \tag{42}$$

$$\partial_t^{(\tau)} u^{(\tau)}(0,\cdot) \quad \text{in } L^2(0,T). \tag{43}$$

Once we have the uniform bound on the time derivative, (42), we deduce from (34) a uniform bound for $u^{(\tau)}$ in $L^2(0,T;H^2(I))$, i.e. (39). Therefore, there exist $u \in L^\infty(0,T;H^1(I)) \cap H^1(0,T;L^2(I))$ and $q \in L^\infty(0,T;\mathscr{W})$ such that (36) and (37) hold. In addition, the compactness result of [31] implies (40). Therefore, since $\bar{f} \in L^\infty(I;C(\mathbb{R}))$ we have $q_x^{(\tau)} = \bar{f}(\cdot,u^{(\tau)}) \to \bar{f}(\cdot,u) = q_x$ uniformly-strongly in $C((0,T];L^\infty(I))$, and then (41). Finally, from (43) we deduce (38). □

End of proof of Theorem 1. We are now ready to pass to the limit $\tau \to 0$. The pair $(u^{(\tau)}, q^{(\tau)})$ satisfies

$$\int_{Q_T} \partial_t^{(\tau)} u^{(\tau)} \xi + \int_{Q_T} u_x^{(\tau)} \xi_x + \int_{Q_T} (u^{(\tau)} q^{(\tau)})_x \xi + \int_0^T \partial_t^{(\tau)} u^{(\tau)}(0,\cdot)\xi(0,\cdot) = 0 \tag{44}$$

for $\xi \in L^2(0,T;H^1(I))$, and

$$q_x^{(\tau)} + \bar{f}(\cdot,u^{(\tau)}) = 0 \quad \text{a.e. in } Q_T, \qquad q^{(\tau)}(d,t) = 0 \quad \text{for all } t \in (0,T]. \tag{45}$$

Taking the limit $\tau \to 0$ in (44)–(45), and using (36)–(41) we obtain that (u,q) satisfies

$$\int_{Q_T} u_t \xi + \int_{Q_T} u_x \xi_x + \int_{Q_T} (uq)_x \xi + \int_0^T u_t(0,\cdot)\xi(0,\cdot) = 0 \tag{46}$$

and

$$q_x + \bar{f}(\cdot,u) = 0 \quad \text{a.e. in } Q_T \quad \text{and} \quad q(d,t) = 0 \quad \text{for all } t \in (0,T].$$

Due to (39) we deduce $u_{xx} \in L^2(Q_T)$. Integrating by parts in (46) and using $\bar{f} \in L^\infty(I, C(\mathbb{R}))$, we deduce that (u,q) satisfies the strong formulation (14)–(18) and it is, therefore, a strong solution of Problem $\bar{\text{P}}$.

Finally, using $\xi := \min\{0, u-m\}$, with $m = \min_{\bar{I}} u_0$, and $\xi := \max\{0, u-1\}$ as test functions in (46) one easily shows that $m \leqslant u \leqslant 1$ in $\bar{Q}_T$. We note at this point that this property implies $\bar{f}(\cdot,u) = f(\cdot,u)$ in $\bar{Q}_T$ and therefore the pair (u,q) is also a strong solution of Problem P.

Finally, if function f and the initial condition satisfy the additional regularity and compatibility conditions stated in Theorem 1 then $u \in C(\bar{Q}_T)$ which implies $uf(\cdot,u), q_x \in C^p(\bar{Q}_T)$ and, therefore, $u_t - u_{xx} \in C^p(\bar{Q}_T)$, implying the additional regularity assertion. □

Proof of Theorem 2. Let (u_1,q_1) and (u_2,q_2) be solutions of Problem P and set $(u,q) := (u_1-u_2, q_1-q_2)$. Then (u,q) satisfies Problem P_D

$$u_t + (uq_1 + u_2 q)_x - u_{xx} = 0 \quad \text{in } Q_\tau,$$

$$q_x + f(x,u_1) - f(x,u_2) = 0 \quad \text{in } Q_\tau,$$

$$u_t(0,\cdot) = u_x(0,\cdot) \quad \text{on } (0,T),$$

$$u_x(d,\cdot) = q(d,\cdot) = 0 \quad \text{on } (0,T),$$

$$u_0 = 0 \quad \text{on } I.$$

We first discuss the case in which f is Lipschitz continuous. Multiplying the first equation of Problem P_D by u and integrating by parts we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(u(0,t)^2 + \int_I u^2\right) + \int_I |u_x|^2 \leqslant \frac{1}{2} q_1(0,t)u^2(0,t) + \frac{1}{2}\int_I f(\cdot,u_1)u^2 \\ + u_2(0,t)|q(0,t)||u(0,t)| + \int_I u_2|q||u_x|. \end{aligned} \tag{47}$$

Using the second equation of Problem P_D, assumption (23) and the continuous embedding $L^2(I) \subset L^1(I)$, we deduce

$$|q(x,t)|^2 \leqslant \left(\int_x^d |f(\cdot,u_1)-f(\cdot,u_2)|\right)^2 \leqslant C_L^2\left(\int_I |u|\right)^2 \leqslant cC_L^2 \int_I u^2, \tag{48}$$

with C_L the constant of Lipschitz of $f(\cdot,s)$ and $c>0$. Using Schwarz's inequality and $\|u_2\|_{L^\infty} \leqslant 1$ we then get from (47)

$$\frac{1}{2}\frac{d}{dt}\left(u(0,t)^2+\int_I u^2\right)+\frac{1}{2}\int_I |u_x|^2 \leqslant c_1 u^2(0,t)+c_2\int_I u^2$$

with $c_1=(\|q_1\|_{L^\infty}+1)/2$ and $c_2=\|f\|_{L^\infty}/2+cC_L^2$. Applying the Lemma of Gronwall with $u_0=0$ we deduce $u=0$ a.e. in Q_T, i.e., $u_1=u_2$. Then, from (48) we also deduce $q_1=q_2$ a.e. in Q_T.

We now let f be a general function satisfying H_1–H_3 and assume that condition (24) holds for u_2. Multiplying the differential equations of Problem P_D by smooth functions φ, ψ satisfying

$$\varphi_t(0,t)+\varphi_x(0,t)=0, \quad \varphi_x(d,t)=\psi(0,t)=0 \quad \text{for any } t\in[0,T], \tag{49}$$

integrating in Q_τ, with $\tau\in(0,T)$, and adding the resulting integral identities we obtain

$$\begin{aligned}u(0,\tau)\varphi(0,\tau)+\int_I u(\cdot,\tau)\varphi(\cdot,\tau) &= \int_{Q_\tau} u(\varphi_t+q_1\varphi_x+\varphi_{xx})-\int_{Q_\tau} q(\psi_x+u_{2x}\varphi)\\ &\quad+\int_{Q_\tau}(f(x,u_1)-f(x,u_2))(u_2\varphi+\psi)\\ &\quad+\int_0^\tau u(0,t)q_1(0,t)\varphi(0,t).\end{aligned} \tag{50}$$

We consider the function defined in Q_τ by

$$h=\frac{f(\cdot,u_1)-f(\cdot,u_2)}{u} \quad \text{if } u\neq 0, \qquad h=0 \quad \text{if } u=0 \tag{51}$$

which is non-positive because $f(x,\cdot)$ is non-increasing and possibly unbounded, since f is not Lipschitz continuous. For $m\in\mathbb{N}$, $m\geqslant 1$, we consider the functions $h^m=T(h)-1/m$, where $T(s)=s$ if $-m<s\leqslant 0$, and $T(s)=-m$ if $s\leqslant -m$. We regularize these functions in such a way that we obtain a smooth sequence $\{h^m\}\subset C^2(Q_\tau)$ satisfying

$$h^{m+1}\leqslant h^m \quad \text{in } Q_\tau, \quad 0>h^m\geqslant -m, \quad h^m\to h \quad \text{a.e. in } Q_\tau.$$

The regularity of solutions of Problem P allows us to introduce sequences $\{q_1^n\}_{n\geqslant 1}$, $\{u_2^n\}_{n\geqslant 1}\subset C^2(Q_T)$ such that, as $n\to\infty$,

$$q_1^n\to q_1 \quad \text{and} \quad u_2^n\to u_2 \quad \text{strongly in } L^2(0,T;H^1(I))\cap C((0,T];C(\bar{I})) \tag{52}$$

and u_2^n satisfying (24). Using these approximations we rewrite (50) as

$$\begin{aligned}u(0,\tau)\varphi(0,\tau)+\int_I u(\cdot,\tau)\varphi(\cdot,\tau) &= \int_{Q_\tau} u(\varphi_t+q_1^n\varphi_x+\varphi_{xx}+h^m(u_2^n\varphi+\psi))\\ &\quad-\int_{Q_\tau} q(\psi_x+u_{2x}^n\varphi)+\int_{Q_\tau} u(h-h^m)(u_2\varphi+\psi)\\ &\quad-\int_{Q_\tau} u_x(q_1-q_1^n)\varphi-\int_{Q_\tau} u(q_{1x}-q_{1x}^n)\varphi\\ &\quad+\int_{Q_\tau} uh^m(u_2-u_2^n)\varphi-\int_{Q_\tau} q(u_{2x}-u_{2x}^n)\varphi\\ &\quad+\int_0^\tau u(0,t)q_1(0,t)\varphi(0,t).\end{aligned} \tag{53}$$

Next we select the functions φ and ψ, being the solutions of

$$\varphi_t + q_1^n\varphi_x + \varphi_{xx} + h^m(u_2^n\varphi + \psi) = 0 \quad \text{in } Q_\tau, \tag{54}$$

$$\psi_x + u_{2x}^n\varphi = 0 \quad \text{in } Q_\tau, \tag{55}$$

$$\varphi(\tau) = \xi \quad \text{on } \Omega, \tag{56}$$

with φ, ψ satisfying (49) and with $\xi \in C^\infty(\bar{I})$.

Lemma 3. *Assume* (24). *Then, for each n and m there exists a unique solution* $\varphi, \psi \in C^{2,1}(\bar{Q}_\tau)$ *of* (54)–(56) *and* (49) *such that* $\|\varphi\|_{L^\infty(Q_\tau)}$ *and* $\|\psi\|_{L^\infty(Q_\tau)}$ *are uniformly bounded with respect to n and m.*

End of proof of Theorem 2. Using the functions provided by Lemma 3 we obtain from (53)

$$\begin{aligned} u(0,\tau)\xi(0) + \int_I u(\cdot,\tau)\xi = & \int_{Q_\tau} u(h - h^m)(u_2\varphi + \psi) - \int_{Q_\tau} u_x(q_1 - q_1^n)\varphi \\ & - \int_{Q_\tau} u(q_{1x} - q_{1x}^n)\varphi + \int_{Q_\tau} uh^m(u_2 - u_2^n)\varphi - \int_{Q_\tau} q(u_{2x} - u_{2x}^n)\varphi \\ & + \int_0^\tau u(0,t)q_1(0,t)\varphi(0,t). \end{aligned} \tag{57}$$

By the uniform estimates from Lemma 3 and (52), we can pass to the limit in (57) and obtain for $n \to \infty$ and then $m \to \infty$

$$u(0,\tau)\xi(0) + \int_I u(\cdot,\tau)\xi = \int_0^\tau u(0,t)q_1(0,t)\varphi(0,t). \tag{58}$$

We choose $\xi = \xi_j \in C^\infty(\bar{I})$ with $\xi_j \to sign(u(\cdot,\tau))$ pointwise and in $L^2(I)$, as $j \to \infty$, where $sign(s) = 1$ if $s \geqslant 0$ and $sign(s) = -1$ if $s < 0$. Then, by Lemma 3

$$u(0,\tau)\xi_j(0) + \int_I u(\cdot,\tau)\xi_j \leqslant c \int_0^\tau |u(0,t)|,$$

with c independent of j. Passing to the limit $j \to \infty$ we deduce

$$|u(0,\tau)| + \int_I |u(\cdot,\tau)| \leqslant c \int_0^\tau |u(0,t)| \tag{59}$$

and Gronwall's Lemma with $u_0 = 0$ implies $u(0,t) = 0$ in $(0,\tau)$. Hence, from (59) we also deduce $u = 0$ in Q_τ, and therefore $u_1 = u_2$ a.e. in Q_τ for any $\tau \in (0,T)$. Checking that this implies $q_1 = q_2$ is straightforward. □

Proof of Lemma 3. Since problem (54)–(56) and (49) is linear with smooth coefficients and data, existence, uniqueness and regularity of solutions is well known, see for instance [22]. We assert that the global maximum of $|\varphi|$, given by $\max|\varphi| = \max\{\max\varphi, -\min\varphi\}$, is attained initially at $t = \tau$. On the contrary, suppose that $\max|\varphi| = \max\varphi$ (positive) is attained at $(x_0, \tau_0) \in \bar{I} \times (0,\tau)$. Then $\varphi_t(x_0,\tau_0) = 0$. If $x_0 \in I$ then we also have $\varphi_x(x_0,\tau_0) = 0$. And, in fact, this is also the case at the boundaries $x = 0$ and $x = d$ due to the boundary conditions (49) satisfied by φ. Then, the φ-equation at (x_0,τ_0) yields

$$\varphi_{xx}(x_0,\tau_0) = -h^m(x_0,\tau_0)(u_{2n}(x_0,\tau_0)\varphi(x_0,\tau_0) + \psi(x_0,\tau_0)). \tag{60}$$

We assert that the right-hand side of (60) is positive, leading then to a contradiction to the assumption of (x_0,τ_0) being a maximum of φ. Suppose the contrary. Then using $h^m < 0$,

$$u_2^n(x_0,\tau_0)\varphi(x_0,\tau_0) + \psi(x_0,\tau_0) \leqslant 0. \tag{61}$$

Integrating the ψ-equation of (55) in $(0,x)$ gives

$$\psi(x,t) = \int_0^x (-u_{2x}^n(y,t))\varphi(y,t)\,\mathrm{d}y. \tag{62}$$

Therefore, from (61), (62) and assumption (24) we obtain

$$
\begin{aligned}
u_2^n(x_0,\tau_0)\varphi(x_0,\tau_0) &\leqslant \int_0^{x_0} u_{2x}^n(y,\tau_0)\varphi(y,\tau_0)\,\mathrm{d}y \\
&\leqslant \int_0^{x_0} |u_{2x}^n(y,\tau_0)|\,\mathrm{d}y \sup_{y\in(0,x_0)} |\varphi(y,\tau_0)| \\
&= \int_0^{x_0} |u_{2x}^n(y,\tau_0)|\,\mathrm{d}y\, \varphi(x_0,\tau_0) < u_2^n(x_0,\tau_0)\varphi(x_0,\tau_0),
\end{aligned}
\tag{63}
$$

a contradiction. Therefore, (60) is non-negative and a maximum of φ cannot be attained at $\tau_0>\tau$. The case of $\max|\varphi| = -\min\varphi$ at $\tau_0>\tau$ is treated in a similar way. Since $\max|\varphi|=0$ is overridden by the initial condition, we deduce that the global maximum of φ must be attained at $t=\tau$, i.e., $\|\varphi\|_{L^\infty}=\|\xi\|_{L^\infty}$, which is independent of m and n. To finish the proof we use (62) and (52) to find $\|\psi\|_{L^\infty(Q_\tau)}\leqslant\|\varphi\|_{L^\infty(Q_\tau)}\|u_2\|_{L^\infty(0,\tau;W^{1,1}(I))}$, which is also independent of m and n. □

Proof of Proposition 1. We first show that $u_x\leqslant L$ in Q_T. Due to the regularity and compatibility assumptions (21)–(22), we may differentiate the first equation of Problem P with respect to x to obtain the following problem for $v:=u_x$

$$
v_t + qv_x - v_{xx} - (2f+uf_u)v = f_x u \quad \text{in } Q_T, \tag{64}
$$

with $v(0,\cdot)=u_t(0,\cdot)$, $v(d,\cdot)=0$ on $(0,T)$, and $v(\cdot,0)=u_{0x}$ on I. By assumption H_1 and (26), we may apply the maximum principle to (64) to deduce that the maximum of v must be non-negative and located on the parabolic boundary. If the maximum is at $x=d$ then we finished, since $v(d,t)=0$. Let us examine the cases in which the maximum is at $x=0$ or at $t=0$. Since, by assumption, $u_{0x}\leqslant L$ in I, we have $v(x,t)\leqslant\max\{\max_{t\in(0,T)}v(0,t),L\}$ in Q_T. Suppose that the maximum of v is attained at $(0,t_0)$. Particularizing the differential equation satisfied by u at $(0,t_0)$ and using the dynamic boundary condition we deduce

$$
u_x(0,t_0)(1+q(0,t_0)) = u(0,t_0)f(0,u(0,t_0)) + u_{xx}(0,t_0).
$$

Since at a maximum of v, $u_{xx}(0,t_0)=v_x(0,t_0)\leqslant 0$, we obtain

$$
v(0,t_0)(1+q(0,t_0))\leqslant u(0,t_0)f(0,u(0,t_0))
$$

and then $v(0,t_0)\leqslant f(0,u_m)\leqslant L$, by (25). Therefore, $u_x=v\leqslant L$ in Q_T.

Define $w(x,t):=u(x,t)-Lx$ for $(x,t)\in Q_T$. Then $w_x\leqslant 0$ in Q_T and $|u_x|\leqslant -w_x+L$. Using (20) and (25) we obtain

$$
\int_0^x |u_x|\leqslant u(0,t)-u(x,t)+2L\leqslant 1-u_m+2L<u_m\leqslant u(x,t) \quad \text{in } Q_T. \qquad \square
$$

Proof of Theorem 3. First, we analyze the case in which $q(0,t_0)=0$ for some $t_0\geqslant 0$. From Eq. (15) and the boundary condition (17) we deduce $\int_I f(\cdot,u(\cdot,t_0))=0$, which implies $u(\cdot,t_0)\equiv 1$ in $[0,1]$ by assumption H_1 and the continuity of u. If the interval $I=(0,d)$ has $d=1$ then we finished. Otherwise, in $(1,d)\times(t_0,\infty)$ function u satisfies the equation $u_t-u_{xx}=0$, the boundary conditions $u(1,t)=1$ and $u_x(d,t)=0$, for $t\geqslant t_0$ and the *initial* data $u(\cdot,t_0)\geqslant 0$, and function q is identically zero. It is then a standard result that $u\to 1$ in $L^2(1,d)$ as $t\to\infty$.

Let us now assume that $q(0,t)>0$ for all $t\geqslant 0$, and let $\eta:[0,T]\to[0,2]$ be given by $\eta(t)=u(0,t)+\int_I u(x,t)\,\mathrm{d}x$. Integrating the u-equation (14) in I and using (20), we find

$$
\eta(T)-\eta(0)=\int_0^T q(0,t)u(0,t)\,\mathrm{d}t > u_m\int_0^T q(0,t) \tag{65}
$$

and therefore

$$
\int_0^\infty q(0,t) \quad \text{is bounded.} \tag{66}
$$

We now obtain uniform estimates in time for u_x and u_t in the norms of $L^\infty((0,\infty);L^2(I))$ and $L^2(Q_\infty)$. Multiplying the u-equation (14) by u, integrating in Q_T and using (15) we get

$$u^2(0,T)+\int_I u^2(\cdot,T)+2\int_{Q_T}|u_x|^2=u_0^2(0)+\int_I u_0^2+\int_{Q_T} f(\cdot,u)(u^2+u^2(0,t))$$
$$\leqslant 2+2\int_{Q_T} f(\cdot,u)=2+2\int_0^T q(0,t).$$

Therefore, estimate (66) implies

$$\|u_x\|_{L^2(Q_\infty)} \quad \text{is bounded.} \tag{67}$$

Multiplying the u-equation by u_t, integrating in Q_T and defining $\phi'(\cdot,s)=sf(\cdot,s)$ with $\phi(\cdot,0)=0$, we get

$$\int_{Q_T}|u_t|^2+\int_0^T|u_t(0,\cdot)|^2+\frac{1}{2}\int_I|u_x(\cdot,T)|^2+\int_I\phi(\cdot,u_0)$$
$$\leqslant\frac{1}{2}\int_I|u_{0x}|^2+\int_I\phi(\cdot,u(\cdot,T))+\frac{1}{2}\|q\|^2_{L^\infty}\|u_x\|^2_{L^2}+\frac{1}{2}\int_{Q_T}|u_t|^2.$$

Therefore, using (67) we deduce

$$\|u_x\|_{L^\infty(0,\infty;L^2(I))},\quad \|u_t\|_{L^2(Q_\infty)}\quad\text{and}\quad\|u_t(0,\cdot)\|_{L^2(0,\infty)}\quad\text{are bounded.} \tag{68}$$

Now we argue as in [21]. Let $\omega(u_0)$ be the ω-limit set of the semi-orbits $u(\cdot,t)$, for $t\geqslant t^*$, given by

$$\omega(u_0)=\{U\in H^1(I):\exists t_n\to\infty \text{ such that } u(\cdot,t_n)\to U \text{ in } L^2 \text{ as } n\to\infty\}.$$

Due to the bound of the gradient (68) the ω-limit set is well defined and non-empty. Let $U=\lim_{n\to\infty}u(\cdot,t_n)$ in $L^2(I)$ and a.e. in I. By the dominated convergence theorem, function Q given by

$$Q(x)=\lim_{n\to\infty}q(x,t_n)=\lim_{n\to\infty}\int_x^d f(\cdot,u(\cdot,t_n))=\int_x^d f(\cdot,U(\cdot)) \tag{69}$$

is well defined for a.e. $x\in I$. Consider the function $U_n(x,s)=u(x,t_n+s)$ for $x\in I$, and $s\in(-1,1)$. We have

$$\int_I|U_n(\cdot,s)-u(\cdot,t_n)|^2\leqslant\int_I\int_{t_n-1}^{t_n+1}|u_t|^2\quad\text{and}\quad|U_n(0,s)-u(0,t_n)|\leqslant\int_{t_n-1}^{t_n+1}|u_t(0,\cdot)|^2.$$

Hence, both

$$\|U_n-u(\cdot,t_n)\|^2_{L^2(I\times(-1,1))}\quad\text{and}\quad\|U_n(0,\cdot)-u(0,t_n)\|^2_{L^2(-1,1)}$$

tend to zero as $n\to\infty$, due to the time derivative bounds (68). Define

$$Q_n(x,s)=q(x,t_n+s)=\int_x^d f(\cdot,u(\cdot,t_n+s))=\int_x^d f(\cdot,U_n(\cdot,s)).$$

By the dominated convergence theorem, we have $Q_n\to Q$ in $L^2(I\times(-1,1))$ and a.e. in $I\times(-1,1)$. Finally, showing that the limit is $(U,Q)=(1,0)$ is standard. Let $\zeta\in C^2(I)$ with $\zeta'(0)=\zeta'(d)=0$, and $\rho\in C_0^2(-1,1)$ such that $\rho\geqslant 0$ and $\int_{-1}^1\rho=1$. Multiplying the u-equation (14) by $\xi(t,x)=\rho(t-t_n)\zeta(x)$, integrating in Q_{t_n+1} and changing to the variable $s=t-t_n$ leads to

$$\int_{-1}^1\left[(\rho'(s)\zeta(0)+\rho(s)\zeta(0)Q_n(0,s))U_n(0,s)+\int_I(\rho'\zeta+\rho\zeta_{xx}+\rho\zeta_x Q_n)U_n\right]=0.$$

Passing to the limit $n\to\infty$ and using the properties of function ρ we find

$$\int_I(\zeta_{xx}U+\zeta_x UQ)+\zeta(0)U(0)Q(0)=0. \tag{70}$$

Choosing $\zeta(x) = 1$, we deduce $U(0)Q(0) = 0$. Since for all $t > 0$ we have $u(0, t) \geqslant u_m > 0$, we deduce $U(0) > 0$, and then $Q(0) = 0$, implying $Q \equiv 0$ in I and $U = 1$ in [0, 1], due to (69) and the properties of f. Finally, since $U \in H^1(I)$ we may integrate by parts in (70) to get, for any $\zeta \in C^2(I)$ with $\zeta'(0) = \zeta'(d) = 0$,

$$\int_I \zeta_x U_x = 0,$$

so U is constant in I, i.e., $U \equiv 1$ in I. □

Proof of Corollary 1. Let (u, q) be a solution of Problem P corresponding to the initial data u_0. By Theorem 3 we have that $u(\cdot, t) \to 1$ in $L^2(I)$ and $u(0, t) \to 1$ pointwise as $t \to \infty$. Therefore, for all $M > 0$ there exists $T_0 < \infty$ such that

$$\int_I (1 - u(\cdot, T_0))^2 + (1 - u(0, T_0))^2 \left(1 + \int_{Q_T} f(\cdot, u)\right) < M.$$

We finish using (29) and Theorem 4 for Problem P with $u_0 := u(\cdot, T_0)$. □

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